Assigning values to δ_i, $1 \leqslant i \leqslant 4$ in (7.6), we can form suitable approximations for the partial derivatives involving the values of u at internal pivots and boundary nodal points. Using these expressions for the derivatives, the differential equation (7.1) can be replaced by a difference equation at internal and boundary pivots.

Another way of replacing the differential equation (7.1) by the difference equation at the pivot (l, m) is as follows. We choose p pivots which are arranged about (l, m) in a definite manner. Take (l, m) pivot and label it 0 and label the rest of the pivots 1, 2, ..., p.

We put

$$Lu_0 = \sum_{i=0}^{p} c_i u_i \tag{7.7}$$

where c_i, $0 \leqslant i \leqslant p$ are arbitrary parameters and u_i is the value of u at the pivot i. For the nodal points in Figure 7.2 and $p=4$, (7.7) becomes

$$Lu_0 = c_0u_0 + c_1u_1 + c_2u_2 + c_3u_3 + c_4u_4 \tag{7.8}$$

With the help of the Taylor series we find

$$u_1 = u(x_l + h\delta_1, y_m)$$
$$= u_0 + h\delta_1 \frac{\partial u_0}{\partial x} + \frac{1}{2}\delta_1^2 h^2 \frac{\partial^2 u_0}{\partial x^2} + R_{01}$$
$$u_3 = u(x_l - h\delta_3, y_m)$$
$$= u_0 - \delta_3 h \frac{\partial u_0}{\partial x} + \frac{1}{2}\delta_3^2 h^2 \frac{\partial^2 u_0}{\partial x^2} + R_{03}$$
$$u_2 = u(x_l, y_m + k\delta_2)$$
$$= u_0 + k\delta_2 \frac{\partial u_0}{\partial y} + \frac{1}{2}k^2\delta_2^2 \frac{\partial^2 u_0}{\partial y^2} + R_{02}$$

and

$$u_4 = u(x_l, y_m - k\delta_4)$$
$$= u_0 - k\delta_4 \frac{\partial u_0}{\partial y} + \frac{1}{2}k^2\delta_4^2 \frac{\partial^2 u_0}{\partial p^2} + R_{04} \tag{7.9}$$

where

$$R_{01} = \frac{\delta_1^3 h^3}{6} \frac{\partial^3 u(\xi_1, y_m)}{\partial x^3}, \; x_l \leqslant \xi_1 \leqslant x_l + h\delta_1$$
$$R_{03} = -\frac{\delta_3^3 h^3}{6} \frac{\partial^3 u(\xi_3, y_m)}{\partial x^3}, \; x_l - h\delta_3 \leqslant \xi_3 \leqslant x_l$$
$$R_{02} = \frac{\delta_2^3 k^3}{6} \frac{\partial^3 u(x_l, \xi_2)}{\partial y^3}, \; y_m \leqslant \xi_2 \leqslant y_m + k\delta_2$$
$$R_{04} = -\frac{\delta_4^3 k^3}{6} \frac{\partial^3 u(x_l, \xi_4)}{\partial y^3}, \; y_m - k\delta_4 \leqslant \xi_4 \leqslant y_m \tag{7.10}$$

partial derivatives of as high an order as we desire. We write at the internal pivot (l, m) as

$$\nabla^2 u(x_l, y_m) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u(x_l, y_m)$$

$$= \left[\frac{4}{h^2}\left(\sinh^{-1}\frac{\delta_x}{2}\right)^2 + \frac{4}{k^2}\left(\sinh^{-1}\frac{\delta_y}{2}\right)^2\right] u(x_l, y_m) \quad (7.19)$$

On the right side of (7.19) we have exact discrete analogue of ∇^2. Thus our difference approximation will then be based upon the polynomial and rational approximations involving the two operators δ_x^2 and δ_y^2.

Consider the approximations

$$4\left(\sinh^{-1}\frac{\delta_x}{2}\right)^2 = \begin{cases} \text{(i)} \ \delta_x^2 + 0(\delta_x^4) \\ \text{(ii)} \ \delta_x^2 - \frac{1}{12}\delta_x^4 + 0(\delta_x^6) \\ \text{(iii)} \ \left(1 + \frac{1}{12}\delta_x^2\right)^{-1}\delta_x^2 + 0(\delta_x^6) \\ \text{(iv)} \ \left(1 + \frac{2}{15}\delta_x^2\right)^{-1}\left(\delta_x^2 + \frac{1}{20}\delta_x^4\right) + 0(\delta_x^8) \end{cases} \quad (7.20)$$

and similar expressions for $[2\sinh^{-1}\delta_y/2]^2$.

The approximation (7.20 i) gives the 5-point formula

$$\nabla^2 u_{l,m} = \left(\frac{1}{h^2}\delta_x^2 + \frac{1}{k^2}\delta_y^2\right) u_{l,m}$$

which may be written as

$$\nabla^2 u_{l,m} = \left[\frac{1}{h^2}H_x^+ + \frac{1}{k^2}H_y^+ - 2\frac{k^2+h^2}{h^2k^2}\right] u_{l,m}$$

or

$$h^2\nabla^2 u_{l,m} = [H_x^+ + \alpha H_y^+ - 2(1+\alpha)] u_{l,m} \quad (7.21)$$

where we denote

$$H_{qx}^+ = E_x^q + E_x^{-q}, \ H_{qy}^+ = E_y^q + E_y^{-q} \quad (7.22)$$

and E is the shift operator.

The approximation (7.20 ii) leads to the 9-point difference scheme

$$\nabla^2 u_{l,m} = \left[\frac{1}{h^2}\left(\delta_x^2 - \frac{1}{12}\delta_x^4\right) + \frac{1}{k^2}\left(\delta_y^2 - \frac{1}{12}\delta_y^4\right)\right] u_{l,m}$$

or

$$h^2\nabla^2 u_{l,m} = \left[\delta_x^2 - \frac{1}{12}\delta_x^4 + \alpha\left(\delta_y^2 - \frac{1}{12}\delta_y^4\right)\right] u_{l,m} \quad (7.23)$$

with order of accuracy $(h^4 + k^4)$.

For square network $h=k$, (7.23) can be simplified to

$$12h^2\nabla^2 u_{l,m} = [-(H_{2x}^+ + H_{2y}^+) + 16(H_x^+ + H_y^+) - 60]\, u_{l,m} \qquad (7.24)$$

The approximation (7.20 iii) gives another 9-point difference scheme

$$\nabla^2 u_{l,m} = \left[\frac{1}{h^2}\left(1+\frac{1}{12}\delta_x^2\right)^{-1}\delta_x^2 + \frac{1}{k^2}\left(1+\frac{1}{12}\delta_y^2\right)^{-1}\delta_y^2\right] u_{l,m}$$

Simplifying we may write it as

$$\left[1+\frac{1}{12}(\delta_x^2+\delta_y^2)\right]\nabla^2 u_{l,m} = \left[\frac{1}{h^2}\delta_x^2 + \frac{1}{k^2}\delta_y^2 + \frac{1}{12}\left(\frac{1}{h^2}+\frac{1}{k^2}\right)\delta_x^2\delta_y^2\right] u_{l,m}$$

or

$$[H_x^+ + H_y^+ + 8]\, h^2\nabla^2 u_{l,m}$$
$$= 20(1+\alpha)\left[-\frac{1}{10}\,\frac{1-5\alpha}{1+\alpha}\,H_y^+ + \frac{1}{10}\,\frac{5-\alpha}{1+\alpha}\,H_x^+ + \frac{1}{20}\,H_x^+H_y^+ - 1\right] u_{l,m} \qquad (7.25)$$

It can be easily verified that the difference schemes (7.21) and (7.25) are the particular cases of the scheme

$$h^2\nabla^2 u_{l,m} = [(1+\sigma\delta_x^2)^{-1}\,\delta_x^2 + \alpha\,(1+\sigma\delta_y^2)^{-1}\,\delta_y^2]\, u_{l,m} \qquad (7.26)$$

where σ is arbitrary. The values $\sigma=0$ and $1/12$ give (7.21) and (7.25) respectively. Simplifying (7.26) and retaining the terms of $0(h^4)$ we get the 9-point scheme

$$[1+\sigma(\delta_x^2+\delta_y^2)]\, h^2\nabla^2 u_{l,m} = [\delta_x^2 + \alpha\delta_y^2 + \sigma(1+\alpha)\,\delta_x^2\delta_y^2]\, u_{l,m}$$

or

$$[\sigma(H_x^+ + H_y^+) + (1-4\sigma)]\, h^2\nabla^2 u_{l,m}$$
$$= [(1-2\sigma(1+\alpha))\, H_x^+ + (\alpha - 2\sigma(1+\alpha))\, H_y^+$$
$$+ \sigma(1+\alpha)\, H_x^+H_y^+ - 2(1+\alpha)(1-2\sigma)]\, u_{l,m} \qquad (7.27)$$

with accuracy of $0(h^2)$.

The neglected term $h^2\delta_x^2\delta_y^2\nabla^2 u_{l,m}/144$ is of high order and does not affect the acccuracy of difference scheme (7.27). For the square network $h=k$, (7.27) becomes

$$[\sigma(H_x^+ + H_y^+) + (1-4\sigma)]\, h^2\nabla^2 u_{l,m}$$
$$= [(1-4\sigma)(H_x^+ + H_y^+) + 2\sigma\, H_x^+ H_y^+ - 4(1-2\sigma)]\, u_{l,m} \qquad (7.28)$$

The approximation (7.20 iv) for $h=k$ gives the 13-point formula

$$h^2\nabla^2 u_{l,m} = \left[\left(1+\frac{2}{15}\delta_x^2\right)^{-1}\left(\delta_x^2+\frac{1}{20}\delta_x^4\right) + \left(1+\frac{2}{15}\delta_y^2\right)^{-1}\right.$$
$$\left.\left(\delta_y^2+\frac{1}{20}\delta_y^4\right)\right] u_{l,m}$$

or

$$\tfrac{1}{2}(2+H_x^+ +H_y^+)\, \nabla^4 u_{l,m}=[36-10\,(H_x^+ +H_y^+]+(H_{2x}^+ +H_{2y}^+)-$$
$$2H_x^+H_y^+ +(H_{2x}^+H_y^+ +H_x^+H_{2y}^+)]\, u_{l,m} \tag{7.36}$$

Alternatively, we write (7.32) as

$$\nabla^4 u\,(x_l,\, y_m)=16h^{-4}\left[\left(\sinh^{-1}\frac{\delta_x}{2}\right)^2+\left(\sinh^{-1}\frac{\delta_y}{2}\right)^2\right]^2 u\,(x_l, y_m) \tag{7.37}$$

and use the approximations (7.20). For instance, if (7.26) is used then we get

$$h^4\nabla^4 u_{l,m}=[(1+\sigma\delta_x^2)^{-1}\,\delta_x^2+(1+\sigma\delta_y^2)^{-1}\,\delta_y^2]^2\, u_{l,m}$$

which on simplification becomes

$$h^4[1+2\sigma\,(\delta_x^2+\delta_y^2)]\,\nabla^4 u_{l,m}$$
$$=[\delta_x^4+2\delta_x^2\delta_y^2+\delta_y^4+4\sigma\,(\delta_x^4\delta_y^2+\delta_y^4\delta_x^2]\, u_{l,m}$$
$$=[4\sigma\,(H_{2x}^+H_y^+ +H_x^+H_{2y}^+)+(1-8\sigma)(H_{2x}^+ +H_{2y}^+)+2(1-16\sigma)$$
$$H_x^+H_y^+ +8(-1+7\sigma)\,(H_x^+ +H_y^+)+4(5-24\sigma)]\, u_{l,m} \tag{7.38}$$

where high order terms have been neglected. The values $\sigma=0$ and $1/12$ give the difference schemes (7.34) and (7.36) respectively.

We shall now discuss the numerical solution of the Laplace and the biharmonic boundary value problems.

7.3 DIRICHLET PROBLEM

Consider the *Laplace* equation in two space dimensions

$$\nabla^2 u=\frac{\partial^2 u}{\delta x^2}+\frac{\partial^2 u}{\partial y^2}=0 \tag{7.39}$$

over the unit square $\mathcal{R}=[(x, y)\colon 0\leqslant x, y\leqslant 1]$, subject to the boundary condition

$$u\,(x, y)=g\,(x, y) \tag{7.40}$$

on the boundary $\partial\mathcal{R}$ of the unit square $\mathcal{R}$.

The statement of the theorem which ensures the existence and uniqueness of the solution of the boundary value problem (7.39)-(7.40) is:

THEOREM (*Maximum Principle*) 7.1 *A nonconstant solution of* (7.39) *in the region* $\mathcal{R}$ *takes its maximum and minimum values on the boundary* $\partial\mathcal{R}$ *of the region* $\mathcal{R}$.

If a uniform mesh of length h in each coordinate direction is imposed on $\mathcal{R}$, then using (7.28), differential equation (7.39) may be approximated at an internal pivot (l, m) by the difference equation

$$[(1-4\sigma)\,(H_x^+ +H_y^+)+2\sigma H_x^+H_y^+ -4\,(1-2\sigma)]\, u_{l,m}=0 \tag{7.41}$$

where σ is a parameter. When $\sigma = 0$ and $1/12$, (7.41) yields the well known five- and nine-point difference approximations of (7.39) which are correct to orders h^2 and h^4 respectively. If $Mh = 1$, the totality of equations (7.41) gives rise to $(M-1)^2$ linear equations in $(M-1)^2$ unknowns of the form

$$\mathbf{Au} = \mathbf{g} \tag{7.42}$$

where **A** is the square matrix of order $(M-1)^2$ and **g** is a vector of order $(M-1)^2$ arising from the boundary values of the problem. The matrix **A** is symmetric for arbitrary σ. We notice from (7.41) that in each row of the matrix **A** every nonzero nondiagonal coefficient will have a sign opposite to that of the diagonal coefficient if $0 \leqslant \sigma < 1/4$. The condition of diagonal dominance is then satisfied and the matrix **A** is irreducible.

Example 7.1 Solve the Dirichlet problem

$$\nabla^2 u = 0 \text{ in } \mathcal{R}$$
$$u = f(x, y) = \log[(x+1)^2 + y^2] \text{ on } \partial\mathcal{R}$$

where $\mathcal{R}$ is the unit square $0 \leqslant x, y \leqslant 1$, with $h = 1/3$.

The region $\mathcal{R}$ is covered with the square network with $h = 1/3$ (Figure 7.3). There are four internal pivots and eight boundary pivots. Furthermore, the boundary pivots of the network lie on the boundary $\partial\mathcal{R}$.

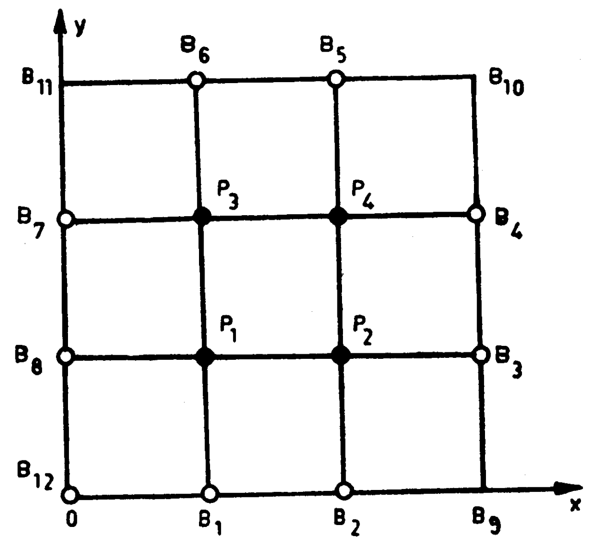

Fig. 7.3 ●-internal and O-boundary pivots

We replace the Laplace equation by the 5-point difference scheme at P_1, P_2, P_3 and P_4. The four simultaneous equations in matrix notation can be written as

$$\begin{bmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -1 & 0 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} f_1+f_8 \\ f_2+f_3 \\ f_6+f_7 \\ f_4+f_5 \end{bmatrix}$$

Using the boundary values, we obtain

$$u_1 = 0.634804, \quad u_2 = 1.059993$$
$$u_3 = 0.798500, \quad u_4 = 1.169821$$

Example 7.2 Solve the Dirichlet problem

$$\nabla^2 u = 0 \text{ in } \mathcal{R}$$
$$u = x^2 + y^2 \text{ on } \partial\mathcal{R}$$

where $\mathcal{R}$ is the semicircle $x^2+y^2 < 1$, $y > 0$ and $\partial\mathcal{R}$ is the boundary of $\mathcal{R}$ with $h = 1/2$.

The theoretical solution is given by

$$u(x, y) = x^2 - y^2 + \sum_{m=0}^{\infty} \frac{-16}{\pi(2m-1)(2m+1)(2m+3)} r^{2m+1} \sin(2m+1)\theta$$

where $x = r\cos\theta$, $y = r\sin\theta$.

We cover the region $\mathcal{R}$ by a square network with $h = 1/2$ (Figure 7.4). Due to the symmetry about y-axis, i.e. $u(-x, y) = u(x, y)$, we solve the Dirichlet problem in the first quadrant only.

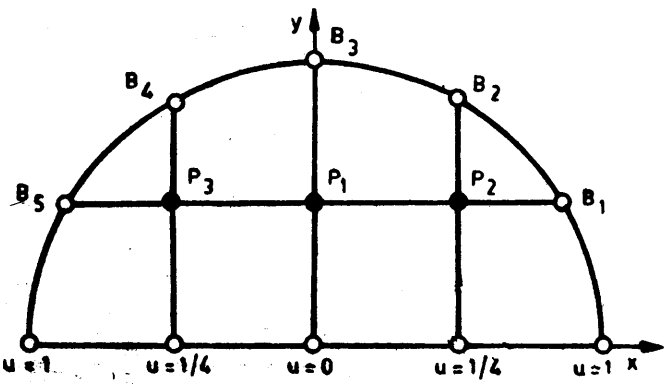

Fig. 7.4 ●-internal and ○-boundary pivots

The Laplace equation is replaced by the 5-point difference scheme (7.30) at the internal boundary pivots. We get the following two equations

for $\quad P_1\left(0, \frac{1}{2}\right), \quad u_1 = \frac{1}{2}u_2 + \frac{1}{4}$

for $$P_2\left(\frac{1}{2}, \frac{1}{2}\right), \quad u_2 = -\frac{\sqrt{3}-1}{2\sqrt{3}}u_1 + \frac{\sqrt{3}+7}{8\sqrt{3}}$$

Thus, we find

$$u_1 = 0.631854, \; u_2 = 0.763708$$

7.4 ITERATIVE METHODS

The system of equations (7.42) is solved by an iteration process which generally can be written as

$$\mathbf{u}^{(n+1)} = \mathbf{H}\mathbf{u}^{(n)} + \mathbf{k} \tag{7.43}$$

where $\mathbf{u}^{(n)}$ denotes the vector of which the elements are the values of $u(x, y)$ at the pivots, obtained after n iterations. The matrix $\mathbf{H}$ is called the *iteration matrix* determined by the iteration method chosen.

If $\mathbf{u}$ is the exact solution of (7.43), then we have

$$\mathbf{u} = \mathbf{H}\mathbf{u} + \mathbf{k} \tag{7.44}$$

which shows that the error $\boldsymbol{\epsilon}^{(n)} = \mathbf{u} - \mathbf{u}^{(n)}$ satisfies

$$\boldsymbol{\epsilon}^{(n+1)} = \mathbf{H}\boldsymbol{\epsilon}^{(n)} \tag{7.45}$$

or

$$\boldsymbol{\epsilon}^{(n)} = \mathbf{H}^n\boldsymbol{\epsilon}^{(0)}$$

It is obvious from (7.45) that the error vector $\boldsymbol{\epsilon}^{(n)}$ obeys the homogeneous form of the iteration equation (7.44). Now, if the sequence of vectors $\mathbf{u}^{(n)}$ is to approach $\mathbf{u}$, then the sequence of vectors $\boldsymbol{\epsilon}^{(n)}$ must approach zero. A necessary and sufficient condition for convergence, i.e. $\lim_{n\to\infty} \boldsymbol{\epsilon}^{(n)} = \mathbf{0}$, with arbitrary starting vector $\boldsymbol{\epsilon}^{(0)}$ is that

$$\lim_{n\to\infty} \mathbf{H}^n = \mathbf{0} \tag{7.46}$$

Let us assume that the matrix $\mathbf{H}$ has a complete set of eigenvectors, say $\mathbf{e}_i$ and corresponding eigenvalues λ_i. Since the eigenvectors are complete, we may expand $\boldsymbol{\epsilon}^{(0)}$ in the form

$$\boldsymbol{\epsilon}^{(0)} = \sum_i a_i\mathbf{e}_i \tag{7.47}$$

Operating successively by $\mathbf{H}$, we get the result

$$\begin{aligned}\boldsymbol{\epsilon}^{(n)} &= \sum_i a_i\lambda_i^n\mathbf{e}_i \\ &= \lambda_1^n\left\{a_1\mathbf{e}_1 + a_2\left(\frac{\lambda_2}{\lambda_1}\right)^n\mathbf{e}_2 + \ldots\right\}\end{aligned} \tag{7.48}$$

Although it is possible for $\boldsymbol{\epsilon}^{(n)}$ to contain no contribution from some particular λ_i, rounding errors will introduce such contributions as the iteration proceeds, and Equation (7.48) may be taken as the general expression for $\boldsymbol{\epsilon}^{(n)}$. Thus, for the error vector to vanish, it is necessary that the modulus of which is smaller than 1 for $\sigma \leqslant 1/4$. For large value of M we obtain

$$\xi = 1 - \frac{\pi^2}{2(1-2\sigma)M^2} \tag{7.66}$$

where we have used

$$\cos\frac{\pi}{M} \approx 1 - \frac{\pi^2}{2M^2}$$

The convergence rate of the iteration method is

$$v = -\log \xi \approx \frac{\pi^2}{2(1-2\sigma)M^2} \tag{7.67}$$

The values $\sigma = 0$ and $\sigma = 1/12$ in (7.67) give the rate of convergence for the well-known 5-point and 9-point formulas.

7.4.2 Gauss-Seidel method

This method is also known as the method of successive displacements, the elements of $\mathbf{u}^{(n+1)}$ replace those of $\mathbf{u}^{(n)}$ in the calculation as soon as they have been computed. We then write the *Gauss-Seidel* method for (7.42) as

$$(\mathbf{L}+\mathbf{D})\mathbf{u}^{(n+1)} + \mathbf{R}\mathbf{u}^{(n)} = \mathbf{g} \tag{7.68}$$

or

$$\mathbf{u}^{(n+1)} = -(\mathbf{L}+\mathbf{D})^{-1}\mathbf{R}\mathbf{u}^{(n)} + (\mathbf{L}+\mathbf{D})^{-1}\mathbf{g} \tag{7.69}$$

For the difference scheme (7.41), the iteration is defined by

$$u_{l,m}^{(n+1)} = \frac{1-4\sigma}{4(1-2\sigma)}[u_{l-1,m}^{(n+1)} + u_{l+1,m}^{(n)} + u_{l,m-1}^{(n+1)} + u_{l,m+1}^{(n)})$$
$$+ \frac{\sigma}{2(1-2\sigma)}[u_{l+1,m+1}^{(n)} + u_{l-1,m+1}^{(n)} + u_{l-1,m-1}^{(n+1)} + u_{l+1,m-1}^{(n+1)}] \tag{7.70}$$

This fixes the matrices $\mathbf{L}$ and $\mathbf{R}$. The error equation of the iteration is written as

$$\epsilon_{l,m}^{(n+1)} = \frac{1-4\sigma}{4(1-2\sigma)}[\epsilon_{l-1,m}^{(n+1)} + \epsilon_{l+1,m}^{(n)} + \epsilon_{l,m-1}^{(n+1)} + \epsilon_{l,m+1}^{(n)}]$$
$$+ \frac{\sigma}{2(1-2\sigma)}[\epsilon_{l+1,m+1}^{(n)} + \epsilon_{l-1,m+1}^{(n)} + \epsilon_{l-1,m-1}^{(n+1)} + \epsilon_{l+1,m-1}^{(n+1)}] \tag{7.71}$$

with the values of $\epsilon_{l,m}^{(n)}$ equal to zero on the boundary $\partial\mathcal{R}$ of the unit square $\mathcal{R}$.

Equation (7.71) can be solved by the method of separation of variables. We put

$$\epsilon_{l,m}^{(n+1)} = \xi\epsilon_{l,m}^{(n)} \tag{7.72}$$

and

$$\epsilon_{l,m}^{(n)} = X_l Y_m \tag{7.73}$$

After substitution of (7.72) and (7.73) in (7.71) and on rearrangement of the right hand side, we find

$$Y_{m+1}-\beta Y_m+\xi Y_{m-1}=0 \tag{7.74}$$

and

$$aX_{l+1}-bX_l+cX_{l-1}=0 \tag{7.75}$$

where

$$a=\frac{(1-4\sigma)}{4(1-2\sigma)}+\frac{\xi\sigma\beta}{2(1-2\sigma)}$$

$$b=\xi-\frac{(1-4\sigma)\beta}{4(1-2\sigma)}$$

$$c=\xi\frac{1-4\sigma}{4(1-2\sigma)}+\frac{\sigma\beta}{2(1-2\sigma)} \tag{7.76}$$

and β is an arbitrary parameter to be determined. The boundary conditions are

$$X_0=0,\ X_M=0,\ Y_0=0,\ Y_M=0 \tag{7.77}$$

The solution of (7.74) subject to (7.77) is

$$Y_m=A_1(\sqrt{\xi})^m\sin\frac{\pi qm}{M},\ 1\leqslant m\leqslant M-1 \tag{7.78}$$

with

$$\beta=2\sqrt{\xi}\cos\frac{q\pi}{M}\text{ and } A_1 \text{ an arbitrary constant.}$$

Similarly, the solution of (7.75) satisfying the boundary conditions is given by

$$X_l=A_2\left(\sqrt{\frac{c}{a}}\right)^l\sin\frac{\pi pl}{M},\ 1\leqslant p\leqslant M-1 \tag{7.79}$$

where A_2 is arbitrary.

Moreover

$$b=2\sqrt{ac}\cos\frac{\pi p}{M} \tag{7.80}$$

By eliminating a, b and c from Equations (7.76) and (7.80), we obtain

$$z^3-4\alpha_1c_q(1+2\alpha_2c_p^2)z^2-4(\alpha_1^2c_p^2-\alpha_1^2c_q^2+4\alpha_2^2c_p^2c_q^2)z-8\alpha_1\alpha_2c_qc_p^2=0 \tag{7.81}$$

where

$$c_p=\cos\frac{\pi p}{M}$$

$$c_q=\cos\frac{\pi q}{M}$$

from which we get

$$z=\frac{wg\pm\sqrt{w^2g^2-4w+4}}{2}$$

Thus, the value $|\xi|=z^2$ is a function of g and w. The largest $|\xi|$ occurs for the largest value of g, which we denote by g_1, and obtain from (7.94), for $p=1$, $q=1$. We then have

$$z=\frac{wg_1+\sqrt{w^2g_1^2-4w+4}}{2} \tag{7.95}$$

for the largest value of z.

We consider the following cases,

(i) $w^2g_1^2-4w+4>0$

$$|z|_{max}=\tfrac{1}{2}(wg_1+\sqrt{(w^2g_1^2-4w+4)})$$
$$<\tfrac{1}{2}(w+\sqrt{(w^2-4w+4)})=\tfrac{1}{2}(w+|w-2|)=1 \text{ if } w<2$$

(ii) $w^2g_1^2-4w+4<0$

$$|z|=\tfrac{1}{2}\sqrt{w^2g_1^2-(w^2g_1^2-4w+4)}=\sqrt{w-1}<1$$

We now study the behaviour of z as a function of w. When the parameter w takes the value unity, the iterative process defined above is the Gauss-Seidel method. The object of using a value for w other than unity is to reduce the largest of the absolute values of the eigenvalues of the matrix H, i.e. ξ. We shall denote this by ξ_{max}. The value of w which minimizes ξ_{max} is called the *optimum accelerating* factor, w_0.

From (7.93) and (7.95), we have

$$\frac{d\xi}{dw}=-\frac{z-g_1z^2}{z-\frac{1}{2}wg_1}$$

At $w=1$, $z=g_1$ or $\xi=g_1^2$ with $g_1<1$ and $d\xi/dw<0$. Therefore, as w increases, ξ decreases until the denominator vanishes, i.e. the slope reaching minus infinity at

$$z=\tfrac{1}{2}w_0g_1 \tag{7.96}$$

From (7.95), we find that the minimum ξ occurs when

$$w_0^2g_1^2=4(w_0-1) \tag{7.97}$$

or

$$w_0=\frac{2(1-\sqrt{1-g_1^2})}{g_1^2}=\frac{2}{1+\sqrt{1-g_1^2}}$$

The value of ξ_{max} is given by

$$\xi_{max}=\frac{w_0^2g_1^2}{4}=w_0-1=\frac{1-\sqrt{1-g_1^2}}{1+\sqrt{1-g_1^2}}$$

where

$$g_1 = \cos\frac{\pi}{M}$$

For large M, we find

$$\xi_{max} = \frac{1-\sin\frac{\pi}{M}}{1+\sin\frac{\pi}{M}} \approx 1-\frac{2\pi}{M} \tag{7.98}$$

The convergence rate is

$$v = -\log\left(1-\frac{2\pi}{M}\right) = \frac{2\pi}{M} \tag{7.99}$$

Hence the convergence rate is much better than it is in the case of the Gauss-Seidel method.

We now state the theorem which enables us to determine the optimal relaxation parameter in the SOR method.

THEOREM 7.2 · Let the real matrix **A** be symmetric, positive-definite, and of the block-tridiagonal form

$$\mathbf{A} = \begin{bmatrix} \mathbf{D}_1 & \mathbf{R}_1 & & & & \\ \mathbf{L}_2 & \mathbf{D}_2 & \mathbf{R}_2 & & & \\ & \mathbf{L}_3 & \mathbf{D}_3 & \mathbf{R}_3 & & \\ & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & \cdot \\ & & & \mathbf{L}_{n-1} & \mathbf{D}_{n-1} & \mathbf{R}_{n-1} \\ & & & & \mathbf{L}_n & \mathbf{D}_n \end{bmatrix}$$

where $\mathbf{D}_i$ are diagonal submatrices. Then $\rho(\mathbf{H}_{GS}) = \rho^2(\mathbf{H}_J)$ and the optimal relaxation parameter w in SOR is given by

$$w_{opt} = \frac{2}{1+(1-\rho(\mathbf{H}_{GS}))^{1/2}}, \quad \rho(\mathbf{H}_{GS}) < 1$$

where $\rho(\mathbf{H}_J)$ is the spectral radius of the Jacobi iteration matrix corresponding to **A**. The optimal value of $\rho(\mathbf{H}_w)$ is

$$\rho(\mathbf{H}_w) = w_{opt} - 1$$

Example 7.3 The system of linear equations

$$\begin{bmatrix} 1 & -a \\ -a & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

The iteration method is convergent only if

$$| \xi | < 1, \quad 1 \leqslant p, q \leqslant M-1$$

The ultimate convergence rate is determined by the maximum magnitude of ξ,

$$\xi_{\max} = \max_{p,q} | \xi | \tag{7.107}$$

From (7.103), the factor ξ lies in the range

$$1 - wf_M \leqslant \xi \leqslant 1 - wf_m \tag{7.108}$$

where f_m and f_M denote the minimum and the maximum values of the function f.

From (7.106), the relationship (7.108) may also be written as

$$1 - w\phi_M \leqslant \xi \leqslant 1 - w(1-4\sigma)\phi_m \tag{7.109}$$

where ϕ_m and ϕ_M denote the minimum and the maximum values of the function ϕ. The smallest and largest values in magnitude of ϕ correspond to $p=q=1$ and to $p=q=M-1$ respectively. Thus, we obtain

$$\xi^* = \max \{ \, | 1 - w\phi_M | \, , \, | 1 - w(1-4\sigma)\phi_m | \, \} \tag{7.110}$$

where, $\phi_m = 8 \sin^2 \pi/2M$ and $\phi_M = 8 \cos^2 \pi/2M$. As w increases from zero, $1 - w(1-4\sigma)\phi_m$ drops slowly from unity, $1 - w\phi_M$ drops rapidly from unity. Thus $\xi^* = 1 - w(1-4\sigma)\phi_m > 0$ so long as $1 - w(1-4\sigma)\phi_m > -(1 - w\phi_M)$. For greater values of w, $\xi^* = -(1 - w\phi_M)$, hence ξ^* then rises with increasing w. The smallest ξ^* occurs where

$$1 - w(1-4\sigma)\phi_m = -(1 - w\phi_M)$$

hence for

$$w = \frac{2}{\phi_M + (1-4\sigma)\phi_m} = \frac{1}{4\left(1 - 4\sigma \sin^2 \dfrac{\pi}{2M}\right)} \tag{7.111}$$

This value of w is denoted by w_{opt}.

The optimum propagating factor is obtained from (7.110) as

$$\xi_{\text{opt}} = \frac{\phi_M - (1-4\sigma)\phi_m}{\phi_M + (1-4\sigma)\phi_m}$$

$$= \frac{1 - 2(1-2\sigma)\sin^2 \dfrac{\pi}{2M}}{1 - 4\sigma \sin^2 \dfrac{\pi}{2M}} \tag{7.112}$$

For large M, (7.112) becomes

$$\xi_{\text{opt}} = 1 - \frac{1}{2}(1-4\sigma)\frac{\pi^2}{M^2} \tag{7.113}$$

The convergence rate is

$$v=\frac{1}{2}(1-4\sigma)\frac{\pi^2}{M^2} \tag{7.114}$$

Example 7.4 Solve the boundary value problem

$$\nabla^2 u=0 \text{ in } \mathcal{R}$$
$$u=e^{3x}\cos 3y \text{ on } \partial\mathcal{R}$$

where $\mathcal{R}=[0 \leqslant x, y \leqslant 1]$, using the 5-point difference scheme with $h=1/3$. Determine the number of iterations required to reduce the error in the solution values by 10^{-6}.

The nodal points are

$$x_l=lh, \quad 0 \leqslant l \leqslant 3$$
$$y_m=mh, \quad 0 \leqslant m \leqslant 3$$

The 5-point difference scheme is given by

$$4u_{l,m}-u_{l+1,m}-u_{l-1,m}-u_{l,m+1}-u_{l,m-1}=0, \quad 1 \leqslant l, m \leqslant 2$$

The boundary conditions become

$$u_{1,0}=2.7183 \quad u_{2,0}=e^2=7.3891$$
$$u_{3,1}=e^3\cos(1)=10.8522$$
$$u_{3,2}=e^3\cos(2)=-8.3585$$
$$u_{0,1}=\cos(1)=0.5403$$
$$u_{0,2}=\cos(2)=-0.4161$$
$$u_{1,3}=e\cos(3)=-2.6911$$
$$u_{2,3}=e^2\cos(3)=-7.3151$$

We have

$l=1, m=1,\ 4u_{1,1}-u_{2,1}-u_{1,2}=3.2586$

$l=2, m=1,\ -u_{1,1}+4u_{2,1}-u_{2,2}=18.2413$

$l=1, m=2,\ -u_{1,1}+4u_{1,2}-u_{2,2}=-3.1072$

$l=2, m=2,\ -u_{2,1}-u_{1,2}+4u_{2,2}=-15.6736$

The above system of equations may be written as

$$\begin{bmatrix} 1 & -1/4 & -1/4 & 0 \\ -1/4 & 1 & 0 & -1/4 \\ -1/4 & 0 & 1 & -1/4 \\ 0 & -1/4 & -1/4 & 1 \end{bmatrix} \begin{bmatrix} u_{1,1} \\ u_{2,1} \\ u_{1,2} \\ u_{2,2} \end{bmatrix} = \begin{bmatrix} 0.8146 \\ 4.5603 \\ -0.7768 \\ -3.9184 \end{bmatrix}$$

The Jacobi iteration method becomes

$$\mathbf{u}^{(n+1)}=\mathbf{H}\mathbf{u}+\mathbf{b}^{(n)}$$

TABLE 7.2 NUMBER OF ITERATIONS REQUIRED FOR THE DIRICHLET PROBLEM WITH $\epsilon = 10^{-6}$

h \ α	0	$\frac{1}{24}$	$\frac{1}{12}$	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{5}{24}$	$\frac{1}{4}$
			Jacobi Method				
$\frac{1}{4}$	45	38	34	28	26	25	23
$\frac{1}{6}$	108	84	76	68	59	50	54
$\frac{1}{8}$	182	139	125	112	98	84	89
			Gauss-Seidel Method				
$\frac{1}{4}$	23	21	19	15	17	16	14
$\frac{1}{6}$	47	44	41	38	34	29	24
$\frac{1}{8}$	78	73	68	63	57	51	44
			SOR Method				
$\frac{1}{4}$	19	17	16	13	13	13	13
$\frac{1}{6}$	35	32	30	27	25	23	22
$\frac{1}{8}$	54	51	47	43	39	35	34

7.5 ALTERNATING DIRECTION METHOD

The alternating direction implicit method developed in Section 5.6.3 may also be applied to the iterative solution of elliptic equations. The basis of the method is to assume that the iteration is analogue to a time variable. The Peaceman-Rachford method for solving the heat flow equation (5.152) is written in (5.188) as

$$\text{(i)} \quad \left(1 - \frac{r}{2}\delta_x^2\right) u_{l,m}^{(n+1/2)} = \left(1 + \frac{r}{2}\delta_y^2\right) u_{l,m}^{(n)}$$

$$\text{(ii)} \quad \left(1 - \frac{r}{2}\delta_y^2\right) u_{l,m}^{(n+1)} = \left(1 + \frac{r}{2}\delta_x^2\right) u_{l,m}^{(n+1/2)} \tag{7.115}$$

where $u_{l,m}^{(n+1/2)}$ is the intermediate value and $r > 0$.

Equation (7.115) is suitable for the numerical calculation but not amenable to the mathematical analysis. We eliminate the intermediate value $u_{l,m}^{(n+1/2)}$ in (7.115) and a relation between $u_{l,m}^{(n+1)}$ and $u_{l,m}^{(n)}$ is obtained as

$$\left(1-\frac{r}{2}\delta_x^2\right)\left(1-\frac{r}{2}\delta_y^2\right)u_{l,m}^{(n+1)}=\left(1+\frac{r}{2}\delta_x^2\right)\left(1+\frac{r}{2}\delta_y^2\right)u_{l,m}^{(n)} \tag{7.116}$$

We now regard the time step in solving (7.116) as one cycle (stage) of iteration. The values $u_{l,m}^{(n)}$ and $u_{l,m}^{(n+1)}$ are the nth and $(n+1)$th iteration approximations to $u_{l,m}$. The starting values $u_{l,m}^{(0)}$ used for the first iteration correspond to the initial iteration. The quantity r is called the iteration parameter.

If $u_{l,m}$ is the exact solution of (7.116), then we have

$$(\delta_x^2+\delta_y^2)u_{l,m}=0 \tag{7.117}$$

which is the 5-point difference approximation to the Lapace equation (7.39). The convergence rate of the iteration procedure (7.116) is determined by studying how the errors decay at successive stages of the iteration.

Using (7.61) and (7.62), we get the propagating factor associated with (7.116) as

$$\xi=\frac{\left(\frac{2}{r}-4\sin^2\frac{\pi p}{2M}\right)\left(\frac{2}{r}-4\sin^2\frac{\pi q}{2M}\right)}{\left(\frac{2}{r}+4\sin^2\frac{\pi p}{2M}\right)\left(\frac{2}{r}+4\sin^2\frac{\pi q}{2M}\right)}$$

$$1\leqslant p,q\leqslant M-1 \tag{7.118}$$

which may also be written as

$$\xi=1-\frac{4r\left(\sin^2\frac{\pi p}{2M}+\sin^2\frac{\pi q}{2M}\right)}{\left(1+2r\sin^2\frac{\pi p}{2M}\right)\left(1+2r\sin^2\frac{\pi q}{2M}\right)}$$

or

$$\xi=1-2\frac{a+b}{(1+a)(1+b)}$$

where

$$a=2r\sin^2\frac{\pi p}{2M}$$

$$b=2r\sin^2\frac{\pi q}{2M}$$

If r is kept constant throughout the iteration then from (7.118) we have

$$|\xi|<1$$

or

$$-1<1-\frac{2(a+b)}{(1+a)(1+b)}<1 \tag{7.119}$$

Thus, the solution of (7.116) converges to the solution of (7.117) as $n \to \infty$ for any positive fixed iteration parameter. From (7.118), we obtain

$$|\xi|_{\max} = \max_{1 \leqslant p \leqslant M-1} \left| \frac{1 - 2r \sin^2 \dfrac{\pi p}{2M}}{1 + 2r \sin^2 \dfrac{\pi p}{2M}} \right| \times \max_{1 \leqslant q \leqslant M-1} \left| \frac{1 - 2r \sin^2 \dfrac{\pi q}{2M}}{1 + 2r \sin^2 \dfrac{\pi q}{2M}} \right|$$

$$= \left[\max_{1 \leqslant p \leqslant M-1} \left| \frac{1 - 2r \sin^2 \dfrac{\pi p}{2M}}{1 + 2r \sin^2 \dfrac{\pi p}{2M}} \right| \right]^2 \tag{7.120}$$

Since $0 < \alpha \leqslant \sin^2 \dfrac{\pi}{2M} < \sin^2 \dfrac{2\pi}{2M} < \dots < \sin^2 \dfrac{\pi(M-1)}{2M} \leqslant \beta$, we try to reduce

$$\max_{1 \leqslant p \leqslant M-1} \left| \frac{1 - 2r \sin^2 \dfrac{\pi p}{2M}}{1 + 2r \sin^2 \dfrac{\pi p}{2M}} \right| \tag{7.121}$$

as small as possible.

To minimize (7.121) as a function of r, consider the function

$$\phi(\lambda; r) = \frac{1 - 2r\lambda}{1 + 2r\lambda}, \; r > 0 \tag{7.122}$$

where $\lambda = \sin^2 \pi p/2M$ and $0 < \alpha \leqslant \lambda \leqslant \beta$.

It is obvious that the derivative of $\phi(\lambda; r)$ with respect to λ is negative for $\lambda \geqslant 0$, so that the maximum value of $|\phi(\lambda; r)|$ occurs at one of the points of the interval $[\alpha, \beta]$. Thus we get

$$\max_{0 < \alpha \leqslant \lambda \leqslant \beta} |\phi(\lambda; r)| = \max \left[\left| \frac{1 - 2r\alpha}{1 + 2r\alpha} \right|, \left| \frac{1 - 2r\beta}{1 + 2r\beta} \right| \right] \tag{7.123}$$

Hence our optimal choice of r should be such that

$$\frac{1 - 2r\alpha}{1 + 2r\alpha} = - \frac{1 - 2r\beta}{1 + 2r\beta}$$

that is,

$$r_{\text{opt}} = 1/2\sqrt{\alpha\beta} = 1 \Big/ \left(2 \sin \frac{\pi}{2M} \cos \frac{\pi}{2M} \right) = \frac{1}{\sin \dfrac{\pi}{M}}$$

from which we get

$$\min_{r>0} \max_{\alpha \leqslant \lambda \leqslant \beta} \left| \frac{1-2r\lambda}{1+2r\lambda} \right| = \frac{1-\tan \dfrac{\pi}{2M}}{1+\tan \dfrac{\pi}{2M}}$$

Hence

$$\xi_{\text{opt}} = \left[\frac{1-\tan \dfrac{\pi}{2M}}{1+\tan \dfrac{\pi}{2M}} \right]^2$$

$$= \left[\frac{1+\cos \dfrac{\pi}{M} - \sin \dfrac{\pi}{M}}{1+\cos \dfrac{\pi}{M} + \sin \dfrac{\pi}{M}} \right]^2$$

$$= \frac{1-\sin \dfrac{\pi}{M}}{1+\sin \dfrac{\pi}{M}} \tag{7.124}$$

which is the same as (7.98). The one parameter Peaceman-Rachford iteration method and the SOR method have the same rate of convergence for solving (7.117).

Let us consider the application of the Mitchell-Fairweather formula for solving (7.39) which may be written in the form

$$\left[1+\left(\sigma-\frac{1}{2}r\right)\delta_x^2\right]\left[1+\left(\sigma-\frac{1}{2}r\right)\delta_y^2\right]u_{l,m}^{(n+1)}$$

$$=\left[1+\left(\sigma+\frac{1}{2}r\right)\delta_x^2\right]\left[1+\left(\sigma+\frac{1}{2}r\right)\delta_y^2\right]u_{l,m}^{(n)} \tag{7.125}$$

where $0 \leqslant \sigma \leqslant 1/4$.

The solution $u_{l,m}$ will satisfy the difference equation

$$(\delta_x^2+\delta_y^2+2\sigma\delta_x^2\delta_y^2)\, u_{l,m}=0$$

which is the 9-point formula (7.28). Equation (7.120) for (7.125) becomes

$$|\xi|_{\max} = \left[\max_{a \leqslant \gamma \leqslant b} \left| \frac{\gamma-r}{\gamma+r} \right| \right]^2 \tag{7.126}$$

where

$$a = \frac{1}{2\cos^2 \dfrac{\pi}{2M}} - 2\sigma$$

$$b = \frac{1}{2\sin^2 \dfrac{\pi}{2M}} - 2\sigma$$

The nodal points are $x_l = lh$, $y_m = mh$, $0 \leqslant l, m \leqslant 3$. The Peaceman-Rachford method for the determination of the intermediate solution values may be written as

$$\begin{aligned} &-u_{l+1,m}^{(n+1/2)} + (\rho+2)u_{l,m}^{(n+1/2)} - u_{l-1,m}^{(n+1/2)} \\ &= u_{l,m-1}^{(n)} + (\rho-2)u_{l,m}^{(n)} + u_{l,m+1}^{(n)} \qquad 1 \leqslant l, m \leqslant 2 \end{aligned}$$

The boundary conditions become

$$\begin{aligned} u_{0,m}^{(n)} &= u_{0,m}^{(n+1/2)} = 0, && 0 \leqslant m \leqslant 3 \\ u_{l,0}^{(n)} &= u_{l,0}^{(n+1/2)} = 0, && 0 \leqslant l \leqslant 3 \\ u_{3,m}^{(n)} &= u_{3,m}^{(n+1/2)} = mh - (mh)^3, && 0 \leqslant m \leqslant 3 \\ u_{l,3}^{(n)} &= u_{l,3}^{(n+1/2)} = (lh)^3 - lh, && 0 \leqslant l \leqslant 3 \end{aligned}$$

We have

$$\begin{aligned} l=1, m=1 \qquad & -u_{2,1}^{(n+1/2)} + (\rho+2)u_{1,1}^{(n+1/2)} - u_{0,1}^{(n+1/2)} \\ & = u_{1,0}^{(n)} + (\rho-2)u_{1,1}^{(n)} + u_{1,2}^{(n)} \\ l=2, m=1 \qquad & -u_{3,1}^{(n+1/2)} + (\rho+2)u_{2,1}^{(n+1/2)} - u_{1,1}^{(n+1/2)} \\ & = u_{2,0}^{(n)} + (\rho-2)u_{2,1}^{(n)} + u_{2,2}^{(n)} \\ l=1, m=2 \qquad & -u_{2,2}^{(n+1/2)} + (\rho+2)u_{1,2}^{(n+1/2)} - u_{0,2}^{(n+1/2)} \\ & = u_{1,1}^{(n)} + (\rho-2)u_{1,2}^{(n)} + u_{1,3}^{(n)} \\ l=2, m=2 \qquad & -u_{3,2}^{(n+1/2)} + (\rho+2)u_{2,2}^{(n+1/2)} - u_{1,2}^{(n+1/2)} \\ & = u_{2,1}^{(n)} + (\rho-2)u_{2,2}^{(n)} + u_{2,3}^{(n)} \end{aligned}$$

The system of equations may be written as

$$\begin{bmatrix} \rho+2 & -1 & 0 & 0 \\ -1 & \rho+2 & 0 & 0 \\ 0 & 0 & \rho+2 & -1 \\ 0 & 0 & -1 & \rho+2 \end{bmatrix} \begin{bmatrix} u_{1,1}^{(n+1/2)} \\ u_{2,1}^{(n+1/2)} \\ u_{1,2}^{(n+1/2)} \\ u_{2,2}^{(n+1/2)} \end{bmatrix}$$

$$= \begin{bmatrix} \rho-2 & 0 & 1 & 0 \\ 0 & \rho-2 & 0 & 1 \\ 1 & 0 & \rho-2 & 0 \\ 0 & 1 & 0 & \rho-2 \end{bmatrix} \begin{bmatrix} u_{1,1}^{(n)} \\ u_{2,1}^{(n)} \\ u_{1,2}^{(n)} \\ u_{2,2}^{(n)} \end{bmatrix} + \frac{8}{27} \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$

or

$$[\rho[\mathbf{I}] + [\mathbf{H}]]\mathbf{u}^{(n+1/2)} = [\rho[\mathbf{I}] - [\mathbf{V}]]\mathbf{u}^{(n)} + \mathbf{b}$$

where

$$[\mathbf{H}]=\begin{bmatrix}\mathbf{B} & \mathbf{0}\\ \mathbf{0} & \mathbf{B}\end{bmatrix},\ [\mathbf{V}]=\begin{bmatrix}2\mathbf{I} & -\mathbf{I}\\ -\mathbf{I} & 2\mathbf{I}\end{bmatrix}$$

$$\mathbf{B}=\begin{bmatrix}2 & -1\\ -1 & 2\end{bmatrix},\ \mathbf{I}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

The Peaceman-Rachford method for the determination of the solution values $u_{l,m}^{(n+1)}$, $1 \leqslant l, m \leqslant 2$ may be written as

$$-u_{l,m-1}^{(n+1)}+(\rho+2)u_{l,m}^{(n+1)}-u_{l,m+1}^{(n+1)}$$
$$=u_{l-1,m}^{(n+1/2)}+(\rho-2)u_{l,m}^{(n+1/2)}+u_{l+1,m}^{(n+1/2)}\quad 1 \leqslant l, m \leqslant 2$$

We have

$$l=1,\ m=1,\qquad -u_{1,0}^{(n+1)}+(\rho+2)u_{1,1}^{(n+1)}-u_{1,2}^{(n+1)}$$
$$=u_{0,0}^{(n+1/2)}+(\rho-2)u_{1,1}^{(n+1/2)}+u_{2,1}^{(n+1/2)}$$
$$l=2,\ m=1,\qquad -u_{2,0}^{(n+1)}+(\rho+2)u_{2,1}^{(n+1)}-u_{2,2}^{(n+1)}$$
$$=u_{1,1}^{(n+1/2)}+(\rho-2)u_{2,1}^{(n+1/2)}+u_{3,1}^{(n+1/2)}$$
$$l=1,\ m=2,\qquad -u_{1,1}^{(n+1)}+(\rho+2)u_{1,2}^{(n+1)}-u_{1,3}^{(n+1)}$$
$$=u_{0,2}^{(n+1/2)}+(\rho-2)u_{1,2}^{(n+1/2)}+u_{2,2}^{(n+1/2)}$$
$$l=2,\ m=2,\qquad -u_{2,1}^{(n+1)}+(\rho+2)u_{2,2}^{(n+1)}-u_{2,3}^{(n+1)}$$
$$=u_{1,2}^{(n+1/2)}+(\rho-2)u_{2,2}^{(n+1/2)}+u_{3,2}^{(n+1/2)}$$

The system of above equations becomes

$$\begin{bmatrix}\rho+2 & 0 & -1 & 0\\ 0 & \rho+2 & 0 & -1\\ -1 & 0 & \rho+2 & 0\\ 0 & -1 & 0 & \rho+2\end{bmatrix}\begin{bmatrix}u_{1,1}^{(n+1)}\\ u_{2,1}^{(n+1)}\\ u_{1,2}^{(n+1)}\\ u_{2,2}^{(n+1)}\end{bmatrix}$$
$$=\begin{bmatrix}\rho-2 & 1 & 0 & 0\\ 1 & \rho-2 & 0 & 0\\ 0 & 0 & \rho-2 & 1\\ 0 & 0 & 1 & \rho-2\end{bmatrix}\begin{bmatrix}u_{1,1}^{(n+1/2)}\\ u_{2,1}^{(n+1/2)}\\ u_{1,2}^{(n+1/2)}\\ u_{2,2}^{(n+1/2)}\end{bmatrix}+\frac{8}{27}\begin{bmatrix}0\\ 1\\ -1\\ 0\end{bmatrix}$$

or

$$[\rho[\mathbf{I}]+[\mathbf{V}]]\mathbf{u}^{(n+1)}=[\rho[\mathbf{I}]-[\mathbf{H}]]\mathbf{u}^{(n+1/2)}+\mathbf{b}$$

$$\mathbf{B} = \begin{bmatrix} 4 & -2 & & & \\ -1 & 4 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 4 & -1 \\ & & & -2 & 4 \end{bmatrix}$$

The matrix equation (7.139) can be solved by the direct or iterative methods.

7.6.1 Derivative condition at the curved boundary

We shall be concerned here with the boundary condition of the form

$$\frac{\partial u}{\partial \mathbf{n}} = A(s) \tag{7.140}$$

prescribed along a part or all of the boundary $\partial\mathfrak{R}$, where s is the boundary parameter and $\mathbf{n}$ stands for the unit normal directed outside the region $\mathfrak{R}$. The region $\mathfrak{R}$ is covered with square net with grid lines parallel to coordinate axes with mesh spacing h.

The part of the boundary $\partial\mathfrak{R}$ along which (7.140) is prescribed and shown in Figure 7.7. The point P_0 is a boundary pivot. If the normal from P_0 to $\partial\mathfrak{R}$ intersects $\partial\mathfrak{R}$ at Q, and also intersects an internal mesh line P_3P_2 at P_1 such that

$$P_3P_1 = h\delta_3,\ P_1P_2 = h\delta_2,\ P_1P_0 = h\delta_1$$

where

$$\delta_2 + \delta_3 = 1,\ 0 \leqslant \delta_1 \leqslant \sqrt{2},\ \delta_2 \geqslant 0$$

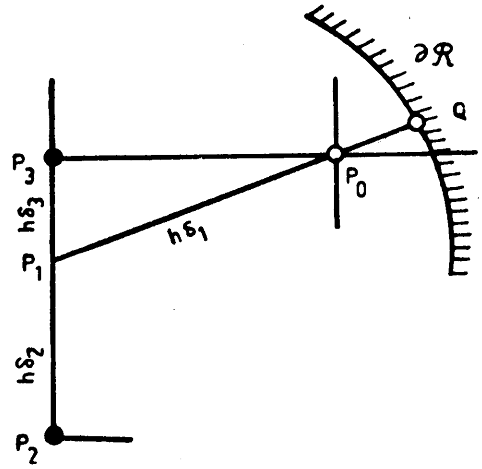

Fig. 7.7 Normal to a curved boundary

then the approximation to $\partial u/\partial n$ at Q can be written as

$$\frac{u_0 - u_1}{h\delta_1} = A(s)$$

$$\frac{1}{h\delta_1}\left[u_0 - \frac{\delta_3 u_2 + \delta_2 u_3}{\delta_2 + \delta_3}\right] = A(s) \qquad (7.141)$$

where s is to be given the value corresponding to the boundary point Q.

7.7 THIRD BOUNDARY VALUE PROBLEM

We consider the third boundary value problem of the form

$$\nabla^2 u = 0$$

over the unit square $\mathcal{R} = [(x, y), 0 \leqslant x, y \leqslant 1]$ with the boundary condition

$$\frac{\delta u}{\partial n} + a(x, y)u = H(x, y) \qquad (7.142)$$

along the boundary $\partial\mathcal{R}$, where $\partial u/\partial n$ is the outwardly directed normal. It can be proved that the boundary value problem will have a unique solution for $a(x, y) > 0$. We cover the square region by a uniform mesh of size h with $Mh = 1$. There are $M+1$ nodal points along each coordinate direction. The Laplace equation is replaced by the 5-point formula

$$(\delta_x^2 + \delta_y^2)u_{l,m} = 0,\ 0 \leqslant l, m \leqslant M \qquad (7.143)$$

Keeping in view (7.135) we replace the boundary conditions (7.142) by the following discretizations

$$u_{1,m} - u_{-1,m} - 2hp_1u_{0,m} = -2bH_{0,m}$$
$$u_{M+1,m} - u_{M-1,m} + 2hq_1u_{M,m} = 2hH_{M,m}$$
$$u_{l,1} - u_{l,-1} - 2hp_2u_{l,0} = -2hH_{l,0}$$
$$u_{l,M+1} - u_{l,M-1} + 2hq_2u_{l,M} = 2hH_{l,M} \qquad (7.144)$$

with

$$p_1 = a_{0,m}, \qquad q_1 = a_{M,m}$$
$$p_2 = a_{l,0}, \qquad q_2 = a_{l,M} \quad \text{and} \quad 0 \leqslant l, m \leqslant M$$

Equations (7.144) are used to eliminate terms in (7.143) which lie outside the region $\mathcal{R}$. Having made the necessary replacements the totality of Equations (7.143) modified by (7.144) gives rise to the matrix equation of the form

$$\mathbf{Au} = 2h\mathbf{G} \qquad (7.145)$$

where $\mathbf{A}$ is a matrix of order $(M+1)^2$ and $\mathbf{G}$ is a column vector of order $(M+1)^2$.

$$\left[\begin{array}{ccc|ccc|ccc}
3 & -1 & & -1 & & & & & \\
-\frac{1}{2} & \frac{5}{2} & -\frac{1}{2} & & -1 & & & & \\
 & -1 & 3 & & & -1 & & & \\
\hline
-\frac{1}{2} & & & \frac{5}{2} & -1 & & -\frac{1}{2} & & \\
 & -\frac{1}{2} & & -\frac{1}{2} & 2 & -\frac{1}{2} & & -\frac{1}{2} & \\
 & & -\frac{1}{2} & & -1 & \frac{5}{2} & & & -\frac{1}{2} \\
\hline
 & & & -1 & & & 3 & -1 & \\
 & & & & -1 & & -\frac{1}{2} & \frac{5}{2} & -\frac{1}{2} \\
 & & & & & -1 & & -1 & 3
\end{array}\right]
\left[\begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \end{array}\right]
=
\left[\begin{array}{c} 0 \\ \frac{3}{4} \\ 2 \\ -\frac{3}{4} \\ 0 \\ \frac{3}{4} \\ -2 \\ -\frac{3}{4} \\ 0 \end{array}\right]$$

mesh size exceeds a certain critical value. If we use the central difference expressions for the derivatives, we get

$$\delta_x^2 u_{l,m} + \delta_y^2 u_{l,m} - R_x(u_{l+1,m} - u_{l-1,m}) - R_y(u_{l,m+1} - u_{l,m-1}) = 0 \qquad (7.147)$$

where $R_x = \dfrac{\bar{u}h}{2\nu}$ and $R_y = \dfrac{\bar{v}h}{2\nu}$.

The numerical solution of (7.147) can be obtained by considering it as the steady state solution of the corresponding time dependent problem. It is easily verified that the difference solution becomes unstable when the mesh size h is such that the parameter

$$R_p = \frac{\bar{V}h}{\nu} > 2 \tag{7.148}$$

where $\bar{V} = \max(|\bar{u}|, |\bar{v}|)$.

The parameter R_p is called the cell *Reynold* or *Peclet number*. The difference scheme (7.147) is of $O(h^2)$. The stability restriction (7.148) is overcome by considering central difference approximations for second order derivatives with backward (upstream) differences for the first order derivatives. We have

$$\delta_x^2 u_{l,m} + \delta_y^2 u_{l,m} - 2R_x \nabla_x u_{l,m} - 2R_y \nabla_y u_{l,m} = 0 \tag{7.149}$$

This scheme is unconditionally stable. The truncation error of (7.149) is given by

$$T_{l,m} = -\frac{\bar{u}h^3}{2\nu}\frac{\partial^2 u(x_l, y_m)}{\partial x^2} - \frac{\bar{v}h^3}{2\nu}\frac{\partial^2 u(x_l, y_m)}{\partial y^2} + O(h^4) \tag{7.150}$$

As in Section 5.3.6, we want to subtract difference quotients $(R_x\delta_x^2 + R_y\delta_y^2)u_{l,m}$ from (7.149) such that the global error of the scheme (7.149) goes from $O(h)$ to $O(h^2)$. Thus, the difference scheme of $O(h^2)$ is given by

$$\delta_x^2 u_{l,m} + \delta_y^2 u_{l,m} - 2R_x\left(\nabla_x u_{l,m} + \frac{1}{2}\delta_x^2 u_{l,m}\right)$$
$$-2R_y\left(\nabla_y u_{l,m} + \frac{1}{2}\delta_y^2 u_{l,m}\right) = 0 \tag{7.151}$$

The difference equations (7.147) or (7.151) may be expressed as (7.42) and solved with the help of an iterative method discussed in Section 7.4.

Alternatively, the Peaceman-Rachford ADI method may be used to obtain the iterative solution of the diffusion convection equation (7.146). We write

$$\text{(i)}\quad \left[1 + 2R_x r\mu_x\delta_x - \frac{1}{2}r\delta_x^2\right]u_{l,m}^{n+1/2} = \left[1 - 2R_y r\mu_y\delta_y + \frac{1}{2}r\delta_y^2\right]u_{l,m}^{n}$$
$$\text{(ii)}\quad \left[1 + 2R_y r\mu_y\delta_y - \frac{1}{2}r\delta_y^2\right]u_{l,m}^{n+1} = \left[1 - 2R_x r\mu_x\delta_x + \frac{1}{2}r\delta_x^2\right]u_{l,m}^{n+1/2} \tag{7.152}$$

Example 7.7 Solve the boundary value problem

$$-\nabla^2 u + \beta\left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right) = 1 \text{ in } \mathcal{R}$$
$$u = 0 \text{ on } \partial\mathcal{R}$$

using the 5-point difference scheme with $h = 1/3$ where $\mathcal{R}$ is the unit square and $\beta > 0$ is a parameter.

The nodal points are $x_l = lh$, $y_m = mh$, $0 \leqslant l, m \leqslant 3$.

The difference scheme with central differences and $R_e = \frac{\beta h}{2}$ may be written as

$$-(\delta_x^2 + \delta_y^2)u_{l,m} + R_e(u_{l+1,m} - u_{l-1,m} - u_{l,m+1} + u_{l,m-1}) = h^2$$

The boundary conditions give

$$u_{0,0} = 0,\ u_{1,0} = 0,\ u_{2,0} = 0\ u_{3,0} = 0$$
$$u_{0,1} = 0,\ u_{0,2} = 0,\ u_{0,3} = 0$$
$$u_{3,1} = 0,\ u_{3,2} = 0,\ u_{3,3} = 0$$
$$u_{1,3} = 0,\ u_{2,3} = 0$$

The difference scheme becomes

$$-(1 + R_e)u_{l-1,m} + 4u_{l,m} - (1 - R_e)u_{l+l,m}$$
$$-(1 - R_e)u_{l,m-1} - (1 + R_e)u_{l,m+1} = 1/9$$

We have, after incorporating the boundary condition:

$$l = 1,\ m = 1,\quad 4u_{1,1} - (1 - R_e)u_{2,1} - (1 + R_e)u_{1,2} = \frac{1}{9}$$

$$l = 2, m = 1,\ -(1 + R_e)u_{1,1} + 4u_{2,1} - (1 + R_e)u_{2,2} = \frac{1}{9}$$

$$l = 1,\ m = 2,\quad 4u_{1,2} - (1 - R_e)u_{2,2} - (1 - R_e)u_{1,1} = \frac{1}{9}$$

$$l = 2, m = 2,\ -(1 + R_e)u_{1,2} + 4u_{2,2} - (1 - R_e)u_{2,1} = \frac{1}{9}$$

The system of equations may be written as

$$\begin{bmatrix} 4 & -(1-R_e) & -(1+R_e) & 0 \\ -(1+R_e) & 4 & 0 & -(1+R_e) \\ -(1-R_e) & 0 & 4 & -(1-R_e) \\ 0 & -(1-R_e) & -(1+R_e) & 4 \end{bmatrix} \begin{bmatrix} u_{1,1} \\ u_{2,1} \\ u_{1,2} \\ u_{2,2} \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

For $R_e = \frac{1}{2}$, we get

$$u_{1,1} = 0.0513,\ u_{2,1} = 0.0662$$
$$u_{1,2} = 0.0406,\ u_{2,2} = 0.0513$$

Using the upwind (downwind) difference schemes for the first derivative, we get

$$-(\delta_x + \delta_y^2)u_{l,m} + 2R_e(\,(u_{l,m} - u_{l-1,m}) - (u_{l,\,m+1} - u_{l,m})\,) = h^2$$

or $$-(1 + 2R_e)u_{l-1,m} + 4(1 + R_e)u_{l,m} - u_{l+1,m} - u_{l,m-1} - (1 + 2R_e)u_{l,m+1} = h^2$$

We have

$$l=1, m=1,\ 4(1+R_e)u_{1,1}-u_{2,1}-(1+2R_e)u_{1,2}=\frac{1}{9}$$

$$l=2, m=1,\ -(1+2R_e)u_{1,1}+4(1+R_e)u_{2,1}-(1+2R_e)u_{2,2}=\frac{1}{9}$$

$$l=1, m=2,\ 4(1+R_e)u_{1,2}-u_{2,2}-u_{1,1}=\frac{1}{9}$$

$$l=2, m=2,\ -(1+2R_e)u_{1,2}+4(1+R_e)u_{2,2}-u_{2,1}=\frac{1}{9}$$

The system of equations may be written as

$$\begin{bmatrix} 4(1+R_e) & -1 & -(1+2R_e) & 0 \\ -(1+2R_e) & 4(1+R_e) & 0 & -(1+2R_e) \\ -1 & 0 & 4(1+R_e) & -1 \\ 4 & -1 & -(1+2R_e) & 4(1+R_e) \end{bmatrix} \begin{bmatrix} u_{1,1} \\ u_{2,1} \\ u_{1,2} \\ u_{2,2} \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

For $R_e=\frac{1}{2}$, we get

$$u_{1,1}=0.0357,\ u_{2,1}=0.0423,$$
$$u_{1,2}=0.0304,\ u_{2,2}=0.0357$$

7.9 AXIALLY SYMMETRIC EQUATION

The axially symmetric equation in cylindrical coordinates (r, ϕ, z) in the region $\mathcal{R}=[0 \leqslant r \leqslant R]\times[0 \leqslant z \leqslant c]$ may be written as

$$\nabla^2 u=\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial z^2}=f(r, z, u) \tag{7.153}$$

with appropriate boundary conditions. The region $\mathcal{R}$, is discretized, using the rectangular network.

$$r_l=lh,\ l=1, 2, \cdots, L$$
$$z_m=mk=msh, m=1, 2, \cdots, M$$

with mesh lengths h and k in r and z directions respectively where

$$s=\frac{k}{h},\ h=\frac{R}{L} \text{ and } k=\frac{C}{M}.$$

A 5-point difference scheme for the Laplace operator ∇^2 at the nodal point (r_l,z_m) may be assumed in the form

$$\nabla^2 u_{l,m}=\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial z^2}\right)_{l,m}$$
$$=-a_0u_{l,m}+a_1u_{l+1,m}+a_3(u_{l,m+1}+u_{l,m-1})+a_5u_{l-1,m} \tag{7.154}$$

The boundary conditions give

$$u_{l,2}=0 \qquad 0 \leqslant l \leqslant 2$$
$$u_{2,m}=0 \qquad 0 \leqslant m \leqslant 2$$

We have

$$l=0,\ m=0,\ -\left(2\left(\frac{2u_{1,0}-2u_{0,0}}{(\frac{1}{2})^2}\right)+\frac{2u_{0,1}-2u_{0,0}}{(\frac{1}{2})^2}\right)=1$$

$$l=1,\ m=0,\ -\left(\frac{0-2u_{1,0}+u_{0,0}}{(\frac{1}{2})^2}+2\,\frac{(0-u_{0,0})}{2\cdot\frac{1}{2}}+\frac{2u_{1,1}-2u_{1,0}}{(\frac{1}{2})^2}\right)=1$$

$$l=0,\ m=1,\ -\left(2\left(\frac{2u_{1,1}-2u_{0,1}}{(\frac{1}{2})^2}\right)+\frac{u_{0,0}-2u_{0,1}}{(\frac{1}{2})^2}\right)=1$$

$$l=1,\ m=1,\ -\left(\frac{u_{0,1}-2u_{1,1}}{(\frac{1}{2})^2}+2\,\frac{0-u_{0,1}}{2(\frac{1}{2})}+\frac{u_{1,0}-2u_{1,1}}{2(\frac{1}{2})^2}\right)=1$$

or

$$\begin{aligned}
24u_{0,0}-16u_{1,0}-8u_{0,1} &= 1\\
-2u_{0,0}+16u_{1,0}-8u_{1,1} &= 1\\
-4u_{0,0}+24u_{0,1}-16u_{1,1} &= 1\\
-4u_{1,0}-2u_{0,1}+16u_{1,1} &= 1
\end{aligned}$$

The above system of equations may be written as

$$\begin{bmatrix} 24 & -16 & -8 & 0\\ -2 & 16 & 0 & -8\\ -4 & 0 & 24 & -16\\ 0 & -4 & -2 & 16 \end{bmatrix}\begin{bmatrix} u_{0,0}\\ u_{1,0}\\ u_{0,1}\\ u_{1,1}\end{bmatrix}=\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$$

which gives

$$u_{0,0}=0.1888 \qquad u_{1,0}=0.1449$$
$$u_{0,1}=0.1516 \qquad u_{1,1}=0.1177$$

7.10 BIHARMONIC EQUATION

Consider the biharmonic equation

$$\nabla^4 u=\frac{\partial^4 u}{\partial x^4}+2\,\frac{\partial^4 u}{\partial x^2\,\partial y^2}+\frac{\partial^4 u}{\partial y^4}=0 \tag{7.159}$$

over the open unit square $\mathscr{R}$, $0 < x, y < 1$, together with the boundary conditions

$$\begin{aligned}
u=\frac{\partial^2 u}{\partial x^2} &=0 && \text{at } x=0,1, && 0<y<1\\
u &=0 && \text{at } y=0,1, && 0<x<1\\
\frac{\partial^2 u}{\partial y^2} &=f(x) && \text{at } y=0, && 0<x<1\\
\frac{\partial^2 u}{\delta y^2} &=g(x) && \text{at } y=1 && 0<x<1
\end{aligned} \tag{7.160}$$

A square mesh is imposed over the region $0 \leqslant x, y \leqslant 1$ with the mesh size h such that $Mh = 1$, $M \geqslant 3$. We use (7.34) to approximate (7.159). The finite difference equations are given by

$$
\begin{aligned}
&[\delta_x^4 + 2\delta_x^2\delta_y^2 + \delta_y^4]u_{l,m} = 0 && 1 \leqslant l, m \leqslant M-1 \\
&\left.\begin{aligned} &u_{l,m} = 0 \\ &u_{l+1,m} = -u_{l-1,m},\ l = 0, M \end{aligned}\right\} && 1 \leqslant m \leqslant M-1 \\
&u_{l,0} = u_{l,M} = 0 \\
&u_{l,1} = -u_{l,-1} + h^2 f_l \\
&u_{l,M+1} = -u_{l,M-1} + h^2 g_l, && 1 \leqslant l \leqslant M-1
\end{aligned}
\qquad (7.161)
$$

where $f_l = f(lh)$, $g_l = g(lh)$ and $u_{l,m}$ is the approximate value of $u(lh, mh)$.

Equation (7.161) represents a system $(M-1)^2$ equations in $(M-1)^2$ unkowns. Alternatively, we may apply the *Conte-Dames* method directly to solve (7.159). Here, the iterative process is given by

$$
\begin{aligned}
u_{l,m}^{(n+1/2)} &= u_{l,m}^{(n)} - r(\delta_x^4 u_{l,m}^{(n+1/2)} + 2\delta_x^2\delta_y^2 u_{l,m}^{(n)} + \delta_y^4 u_{l,m}^{(n)}) \\
u_{l,m}^{(n+1)} &= u_{l,m}^{(n+1/2)} - r\delta_y^4 u_{l,m}^{(n+1)} + r\delta_y^4 u_{l,m}^{(n)}
\end{aligned}
\qquad (7.162)
$$

where $u_{l,m}^{(0)}$ is an initial approximation to $u_{l,m}$, $1 \leqslant l, m \leqslant M-1$ and r is an iteration parameter chosen to accelerate convergence. If $u_{l,m}^{(n+1/2)}$ is eliminated, Equations (7.162) become

$$
\begin{aligned}
u_{l,m}^{(n+1)} = u_{l,m}^{(n)} &- r(\delta_x^4 u_{l,m}^{(n+1)} + 2\delta_x^2\delta_y^2 u_{l,m}^{(n)} + \delta_y^4 u_{l,m}^{(n+1)}) \\
&- r^2\delta_x^2\delta_y^2 (u_{l,m}^{(n+1)} - u_{l,m}^{(n)})
\end{aligned}
$$

which may be written as

$$
\begin{aligned}
&(1 + r\,\delta_x^4)(1 + r\,\delta_y^4)\,u_{l,m}^{(n+1)} \\
&= [(1 + r\,\delta_x^4)(1 + r\,\delta_y^4) - r\,(\delta_x^4 + 2\delta_x^2\delta_y^2 + \delta_y^4)]\,u_{l,m}^{(n)}
\end{aligned}
\qquad (7.163)
$$

The exact solution $u_{l,m}$ of (7.163) will satisfy

$$
(\delta_x^4 + 2\delta_x^2\delta_y^2 + \delta_y^4)\,u_{l,m} = 0
$$

Thus, the formula (7.162) constitutes a convergent iterative method for solving the biharmonic equation (7.159).

Example 7.9 Solve the boundary value problem

$$
\nabla^4 u = 1, \qquad 0 \leqslant x, y \leqslant 1
$$

subject to the boundary condition (see Figure 7.10)

$$
\begin{aligned}
u = 0,\ \frac{\partial u}{\partial x} &= 0, && \text{on } AD \\
u = 0,\ \frac{\partial u}{\partial y} &= 0, && \text{on } AB \\
u = 0,\ \frac{\partial u}{\partial x} &= 0, && \text{on } BC
\end{aligned}
$$

$$u=0, \quad \frac{\partial u}{\partial y}=-1, \quad \text{on } CD$$

Use the 13-point scheme with $h=\frac{1}{3}$.

The nodal points are $x_l=lh$, $y_m=mh$, $0 \leqslant l, m \leqslant 3$. The square region $0 \leqslant x, y \leqslant 1$, is extended such that the boundary pivots may now be considered as the internal points of the extended network as shown in Figure 7.10.

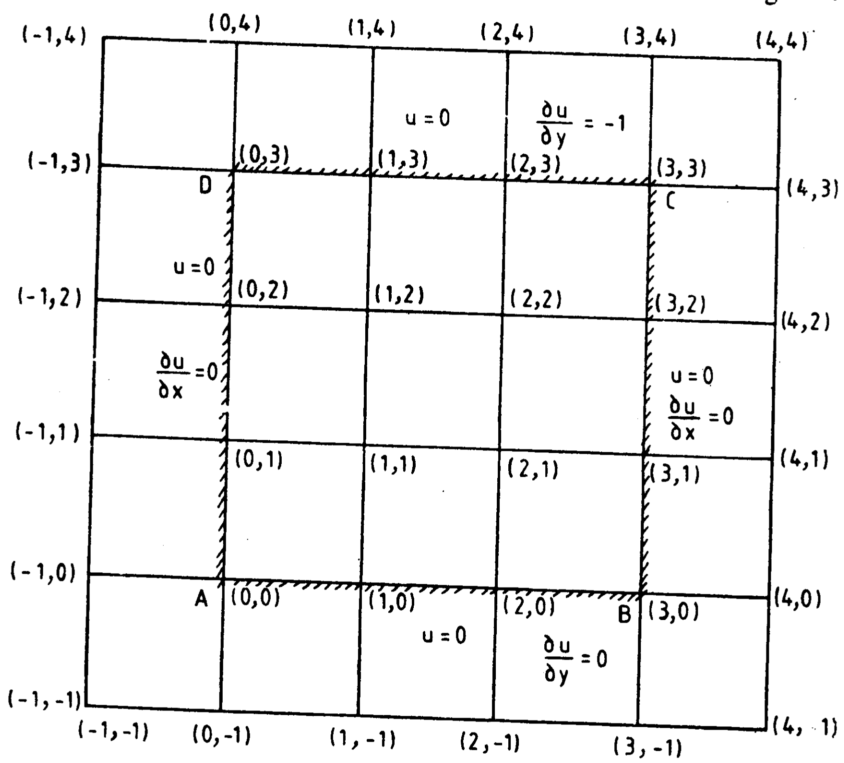

Fig. 7.10 Internal and boundary pivots for biharmonic boundary value problem

The 13-points difference scheme in schematic form may be written as

$$\nabla^4 u_{l,m}=\frac{1}{h^4}\left(\begin{array}{c|c|c|c|c} & & 1 & & \\ \hline & 2 & -8 & 2 & \\ \hline 1 & -8 & 20 & -8 & 1 \\ \hline & 2 & -8 & 2 & \\ \hline & & 1 & & \end{array}\right) u_{l,m}$$

The boundary conditions become

$$u_{0,0}=0 \qquad u_{1,0}=0 \qquad u_{2,0}=0 \qquad u_{3,0}=0$$

$$u_{0,1}=0 \qquad u_{0,2}=0 \qquad u_{0,3}=0$$

$$u_{3,1}=0 \qquad u_{3,2}=0 \qquad u_{3,3}=0$$

$$u_{1,3}=0 \qquad u_{2,3}=0$$

$$u_{1,1}-u_{1,-1}=0 \qquad u_{2,1}-u_{2,-1}=0$$

$$u_{1,1}-u_{-1,1}=0 \qquad u_{1,2}-u_{-1,2}=0$$

$$u_{4,1}-u_{2,1}=0 \qquad u_{4,2}-u_{2,2}=0$$

$$\frac{u_{1,4}-u_{1,2}}{2h}=-1 \qquad \frac{u_{2,4}-u_{2,2}}{2h}=-1$$

We have

$$l=1,\ m=1,\ [(u_{-1,1}+u_{3,1}+u_{1,3}+u_{1,-1})+2(u_{0,0}+u_{2,0}+u_{2,2}+u_{0,2})$$
$$-8(u_{2,1}+u_{1,2}+u_{0,1}+u_{1,0})+20u_{1,1}]=\frac{1}{81}$$

$$l=2,\ m=1,\ [(u_{4,1}+u_{2,3}+u_{0,1}+u_{2,-1})+2(u_{1,0}+u_{3,0}+u_{3,2}+u_{1,2})$$
$$-8(u_{3,1}+u_{2,2}+u_{1,1}+u_{2,0})+20u_{2,1}]=\frac{1}{81}$$

$$l=1,\ m=2,\ [(u_{-1,2}+u_{1,4}+u_{3,2}+u_{1,0})+2(u_{0,3}+u_{2,3}+u_{0,1}+u_{2,1})$$
$$-8(u_{0,2}+u_{2,2}+u_{1,3}+u_{1,1})+20u_{1,2}]=\frac{1}{81}$$

$$l=2,\ m=2,\ [(u_{0,2}+u_{2,4}+u_{4,2}+u_{2,0})+2(u_{1,1}+u_{3,1}+u_{3,3}+u_{1,3})$$
$$-8(u_{1,2}+u_{3,2}+u_{2,3}+u_{2,1})+20u_{2,2}]=\frac{1}{81}$$

Substituting the boundary conditions, we get

$$22u_{1,1}-8u_{2,1}-8u_{1,2}+2u_{2,2}=\frac{1}{81}$$

$$-8u_{1,1}+22u_{2,1}+2u_{1,2}-8u_{2,2}=\frac{1}{81}$$

$$-8u_{1,1}+2u_{2,1}+22u_{1,2}-8u_{2,2}=\frac{55}{81}$$

$$2u_{1,1}-8u_{2,1}-8u_{1,2}+22u_{2,2}=\frac{55}{81}$$

which may be written as

$$\begin{bmatrix} 11 & -4 & -4 & 1 \\ -4 & 11 & 1 & -4 \\ -4 & 1 & 11 & -4 \\ 1 & -4 & -4 & 11 \end{bmatrix} \begin{bmatrix} u_{1,1} \\ u_{2,1} \\ u_{1,2} \\ u_{2,2} \end{bmatrix} = \frac{1}{162} \begin{bmatrix} 1 \\ 1 \\ 55 \\ 55 \end{bmatrix}$$

Solving we obtain

$$u_{1,1}=0.0265,\ u_{2,1}=0.0265,\ u_{1,2}=0.0599$$

$$u_{2,2}=0.0599$$

Bibliographical Note

The rigorous treatment of the difference methods for the elliptic equations is given in 86 and 102. A very readable discussion is found in 9, 114, 184 and 207. The high order difference schemes for the Laplace and biharmonic operators are available in 46. The third boundary value problem for the Laplace equation is discussed in 120.

The theory and practice of iterative methods for the solution of the elliptic equation together with an extensive bibliography is included in 251 and 253. The iterative methods based on 9-point difference formula for Laplace's equation are considered in 81 and 249. The three level alternating direction implicit iterative methods for solving the Dirichlet problem are studied in 30, 108 and 133. The A.D.I. iterative method for the solution of the biharmonic equation is given in 49.

Problems

1. The function $u(x, y)$ satisfies the differential equation

$$\nabla^2 u+\frac{1}{2}=0$$

for $|x| \leqslant 1$, $|y| \leqslant 1$, and the boundary condition $u(x, y)=0$ for $|x|=1$, $|y|=1$. Determine an approximate value of $u(0, 0)$, using the 5-point difference approximation. Use $h=1$ and $h=\frac{1}{2}$, and perform the Richardson extrapolation.
2. Solve the differential equation $\nabla^2 u+1=0$ by using the 5-point dffer-ence scheme. The geometry and the boundary conditions are specified in the Figures 7.11, 7.12 and 7.13
3. Solve the differential equation $\nabla^2 u=16$ on a square of side 2 with $u=0$ on the boundary. Use the 5-point formula.
   (i) Formulate the corresponding difference equation with mesh-size h in both directions.
   (ii) Solve the difference equation exactly for $h=1$ and $h=1/2$.
   (iii) Give the formulas for a convergent iterative procedure for the solution of the difference equation.
   (iv) Give the formulas for the solution by successive over-relaxation method. (BIT 4 (1964), 61)
4. If $\nabla^2 u=0$ in the region $\mathcal{R}$ lying out side the square $|x|=1, |y|=1$ and inside the square $|x|=2, |y|=2$ and if $u=0$ along the outer boundary and $u=300$ along the inner boundary of $\mathcal{R}$, obtain approximate